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Interaction of N atoms with the radiation field in the restricted rotating-wave approximation: I. General theory

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Abstract. A quantum model of N two-level atoms coupled by a single-mode radiation field under a restricted rotating-wave approximation is studied, assuming a dipole-interaction approximation and that the total number of excitations of the system is conserved. A solution is found for the case of spontaneous emission when the atoms are initially prepared in the state of complete inversion. This solution is valid for all times. A general solution of the model is also found for any initial configuration of the system, in the cases of both spontaneous and stimulated emission. This solution is in a closed form and valid for times $\tau < \tau_{max}$.

1. Introduction

The model of two-level systems interacting via the electromagnetic field has been a subject of considerable study over the last two decades.

Dicke (1954) proposed that atoms under special preparation could radiate in a collective way at a rate proportional to the square of the number of atoms. He called this phenomenon 'super-radiance'.

The problem was solved exactly (in the rotating-wave approximation) for one atom (Jaynes and Cummings 1963). Several attempts have been made to extend this solution for N atoms and although no exact solution was found in the past, many workers have found exact or approximate numerical solutions for both fully quantum and semi-classical models (Jaynes and Cummings 1963; Tavis and Cummings 1967, 1968, 1969; Eberly 1968; Eberly and Rehler 1969, 1970; Mallory 1969; Scharf 1970; Bonifacio and Preparata 1970; Arecchi and Courtens 1970; Walls and Barakat 1970; Senitzky 1970; Bonifacio *et al* 1971a, b; Rehler and Eberly 1971; Leonardi *et al* 1972; Thompson 1972; Eberly 1972; Narducci *et al* 1973a, b, c; Argawal 1973; Morawitz 1973; Smithers and Lu 1974; Narducci *et al* 1974; McGillivray and Feld 1976; Glauber and Haake 1976; Ressayre and Tallet 1977; Lee 1977; Orszag 1977).

In this paper, no loss of memory is assumed (the Markovian assumption), and no perturbation expansions are made. The usual dipole interaction and restricted rotating-wave approximations are made. The only additional approximation made turns out to be insignificant for a reasonably large number of atoms.

For the first time a closed analytic solution is found for the model. The model carries the following physical assumptions:

(a) The dipole interaction is a good representation of the physical system.

(b) The operator (a^+a+R_3) is conserved, which in physical terms simply means that the number of excitations of the atoms and radiation field is conserved. This last assumption is a restricted version of the rotating-wave approximation (RWA), a feature present in almost all previous work.

Although some particular cases are compared with existing work, the detailed numerical analysis and discussion associated with the behaviour of the solution will be presented in a future publication (Orszag 1979a).

In § 2 the model is briefly reviewed for the off-resonance case with one single radiation mode. In § 3 a solution is found for the case $M = r$, valid for all times, as well as a general solution for any initial configuration of the system, valid up to a maximum time τ_{max}. Both solutions correspond to the case of spontaneous emission ($n(0) = 0$).

In § 4 some particular cases are studied and compared with previous work. The original Dicke results are retrieved for the short-time single photon emission ($n = 1$). In § 5 the results are generalised for stimulated emission, again valid up to a maximum time τ_{max}. Finally, the accuracy of the present calculations as well as other mathematical aspects of the theory are discussed at the end (§ 6).

2. The model

If we consider a system of N two-level atoms interacting via the radiation field (i.e. any other interaction mechanism is discarded in this work) and assuming that the geometry or size of the system is such that only one mode of the field is necessary to describe it, the Hamiltonian of the system can be written as

$$H = \hbar a^+ a(\omega) + \hbar\omega_0 R_3 + \hbar K(a + a^+)(R^+ + R^-), \tag{1}$$

where R_3, R^+, R^- are respectively the z component, raising and lowering operators for the atoms in the angular momentum representation, a and a^+ are the annihilation and creation operators for the field, K is the coupling constant, ω the frequency of radiation and ω_0 the separation of the two energy levels in units of $\hbar$. In resonance $\omega = \omega_0$.

Apart from the above assumptions we further assume that the emitting medium has a low density such that the spatial part of the wavefunctions of the individual atoms do not overlap and symmetrisation is not necessary. Let us define

$$\hat{N} = a^+ a + R_3. \tag{2}$$

Then it is simple to prove that

$$[\hat{N}, H] = 2\hbar K(a^+ R^+ - aR^-). \tag{3}$$

We will assume that N is conserved or that

$$[\hat{N}, H] = 0. \tag{4}$$

Notice that assumption (4) implies that the difference of the two terms, normally neglected in the rotating-wave approximation, is zero; therefore this is a weaker condition than the rotating-wave approximation (RWA). Dividing equation (1) by $\hbar K$, it can be written as

$$H/\hbar K = \omega_1 \hat{N} + R_3 \Delta_1 + (a + a^+)(R^+ + R^-) \tag{5}$$

where

$$\hat{N} = a^+ a + R_3, \qquad \omega_1 = \omega/K, \qquad \Delta_1 = \Delta/K = (\omega_0 - \omega)/K. \tag{6}$$

A convenient representation for the atomic field operators can be constructed as a direct product of the photon-number eigenstates and Dicke states

$$|n\rangle|r, m\rangle.$$

According to the well-known angular momentum algebra

$$R^2|r, m\rangle = r(r+1)|r, m\rangle,$$
$$R_3|r, m\rangle = m|r, m\rangle. \tag{7}$$

Since the angular momentum operator R^2 also commutes with the Hamiltonian, the quantum numbers r and $M = n + m$ are conserved. If an initial state $|r, m(0) = M\rangle|n(0) = 0\rangle$ is assumed (to simplify the notation we will write $|M\rangle|0\rangle$), then the probability amplitude for n photons at the scaled time can be written as

$$p(n, \tau) = \langle n|\langle M - n| \exp\{-\mathrm{i}\tau[\omega_1\hat{N} + \Delta_1 R_3 + (a + a^+)(R^+ + R^-)]\}|M\rangle|0\rangle \tag{8}$$

where $\tau = Kt$, t being the real time.

3. The solution (spontaneous emission)

As a first step we shall attempt to factorise the exponential appearing in equation (8), making use of a very ingenious set of 'unscrambling' theorems derived by Arecchi *et al* (1972). They have shown that for any representation of the rotational group algebra

$$\exp(\omega_+ R^+ + \omega_- R^- + \omega_z R_3) = \begin{pmatrix} \cosh k + \frac{\omega_z}{2}\frac{\sinh k}{k} & \omega_+\frac{\sinh k}{k} \\ \omega_-\frac{\sinh k}{k} & \cosh k - \frac{\omega_z \sinh k}{2k} \end{pmatrix} \tag{9}$$

where

$$k = \left(\omega_+\omega_- + \frac{\omega_z^2}{4}\right)^{1/2}. \tag{10}$$

They also showed that, for example,

$$\exp(y_- R^-) \exp(\ln y_z)R_3 \exp(y_+ R^+) = \begin{pmatrix} (y_z)^{1/2} & (y_+)(y_z)^{1/2} \\ (y_-)(y_z)^{1/2} & (y_z)^{-1/2} + (y_+ y_-)y_z^{1/2} \end{pmatrix}. \tag{11}$$

Equations (9), (10) and (11) enable us to identify each matrix element and therefore factorise equation (8). The quantities to be determined are y_+, y_- and y_z. From the four equations obtained by the identification only three are independent since the determinant of these matrices is unity (Arecchi *et al* 1972). Setting now (according to equation (8))

$$\omega_z = -\mathrm{i}\Delta_1\tau$$

which is a c number, and

$$\omega_+ = \omega_- = -\mathrm{i} \tan \tau(a + a^+), \tag{12}$$

which are operators, the determinant of the matrix is still unity provided we consider a function of $(a^+ + a)$ as its power series expansion and bearing in mind that ω_+ and ω_-

commute, so that the order of the factors is unimportant when calculating k. Therefore k becomes

$$k = \mathrm{i}\tau[(a+a^+)^2+\Delta_1^2/4]^{1/2} = \mathrm{i}K_1 \tag{13}$$

and the hyperbolic functions cosh K and sinh K become cos K_1 and i sin K_1, and y_z, $y_\pm$ are easily determined as

$$\begin{aligned} y_z &= [\cos K_1 - \mathrm{i}(\Delta_1\tau/2)(\sin K_1)/K_1]^2, \\ y_+ = y_- &= \frac{-\mathrm{i}\tau(a+a^+)\,(\sin K_1)/K_1}{\cos K_1 - \mathrm{i}\Delta_1\tau\,(\sin K_1)/2K_1}. \end{aligned} \tag{14}$$

From equations (9), (11) and (4), we can write

$$\begin{aligned} &\exp\{-\mathrm{i}\tau[\Delta_1 R_3 + (a+a^+)(R^+ + R^-) + \omega_1\hat{N}]\} \\ &\quad = \exp(-\mathrm{i}\tau\omega_1\hat{N})\exp(y_- R^-)\exp[(\ln y_z)R_3]\exp(y_+R^+) \end{aligned} \tag{15}$$

where $y_\pm$ and y_z are given by equation (14). Making use of our result (equation (15)), the probability amplitude $p(n, \tau)$ can now be written as

$$p(n,\tau) = \exp(-\mathrm{i}\omega_1 M\tau)\langle n|\langle M-n|\exp(y_-R^-)\exp[(\ln y_z)R_3]\exp(y_+R^+)|M\rangle|0\rangle. \tag{16}$$

Given the initial state $|M\rangle|0\rangle$, the Hilbert space of our working basis is spanned by $(M+r+1)$ linearly independent states, since $M \leqslant r$. The states are

$$|M\rangle|0\rangle;\ |M-1\rangle|1\rangle;\ \ldots\ |-r\rangle|M+r\rangle. \tag{17}$$

For the case of stimulated emission $(n(0) \neq 0)$ the conserved quantity is $M = m(0)+n(0)$ and the dimension of the space is again $(M+r+1)$ if $M \leqslant r$ and $(2r+1)$ if $M > r$ (Narducci *et al* 1973b).

Proceeding with the calculation of $p(n, \tau)$ we write

$$p(n,\tau) = \exp(-\mathrm{i}\omega_1 M\tau)\langle n|\langle M-n|\exp(y_-R^-)(y_z)^M|M\rangle|0\rangle. \tag{18}$$

To derive equation (18), we used the following facts:

$$\begin{aligned} &\exp(y_+R^+)|M\rangle|0\rangle = |M\rangle|0\rangle, \\ &\exp[(\ln y_z)R_3]|M\rangle|0\rangle = (y_z)^M|M\rangle|0\rangle. \end{aligned} \tag{19}$$

Expanding $\exp(y_-R^-)$ in equation (18) and using the property of the angular momentum operator R^-

$$(R^-)^n|M\rangle = \left[\prod_{q=1}^{n}(r+M-q+1)(r-M+q)\right]^{1/2}|M-n\rangle \tag{20}$$

we obtain

$$p(n,\tau) = \frac{\exp(-\mathrm{i}\omega_1 M\tau)}{n!}\left[\prod_{q=1}^{n}(r+M-q+1)(r-M+q)\right]^{1/2} M_{n0}, \tag{21}$$

where

$$M_{n0} \equiv \langle n|(y_-)^n(y_z)^M|0\rangle. \tag{22}$$

Taking the resonant case $\Delta_1 = 0$, the parameters of equation (14) become

$$K_1 = \tau(a + a^+), \qquad y_z = \cos^2 \tau(a + a^+),$$
$$y_+ = y_- = -\mathrm{i} \tan \tau(a + a^+), \tag{23}$$

and

$$M_{n0} = \langle n | \tan^n \tau(a + a^+) \cos^{2M} \tau(a + a^+) |0\rangle (-\mathrm{i})^n. \tag{24}$$

The calculation of the probability amplitude $p(n, \tau)$ has been reduced to the computation of the matrix element M_{n0}.

Let us first solve the special case when

(*a*) $2M \geqslant n$. In this case we write

$$M_{n0} = [(-\mathrm{i})^{2n}/2^{2M}] \langle n | \{\exp[\mathrm{i}\tau(a + a^+)] - \exp[-\mathrm{i}\tau(a + a^+)]\}^n \{\exp[\mathrm{i}\tau(a + a^+)] + \exp[-\mathrm{i}\tau(a + a^+)]\}^{2M-n} |0\rangle. \tag{25}$$

M_{n0} can finally be written as

$$M_{n0} = \frac{(-\mathrm{i})^{2n}}{2^{2M}} \langle n | \sum_{s_1=0}^{n} \sum_{s_2=0}^{2M-n} \binom{n}{s_1} \binom{2M-n}{s_2} (-1)^{n-s_1} \exp[2\mathrm{i}\tau(a + a^+)(s_1 + s_2 - M)] |0\rangle. \tag{26}$$

Let us define

$$\alpha = 2\mathrm{i}\tau(s_1 + s_2 - M); \tag{27}$$

then, using the Baker–Campbell–Hausdorff (BCH) formula (Wilcox 1967), we can write

$$\exp[\alpha(a + a^+)] = \exp(\alpha a^+) \exp(\alpha a) \exp(\alpha^2/2). \tag{28}$$

Making use of equation (28), M_{n0} can be written as

$$M_{n0} = \frac{(-\mathrm{i})^{2n}}{2^{2M}} \langle n | \sum_{s_1;s_2} \exp(\alpha a^+) \exp(\alpha a) \exp(\alpha^2/2) \binom{n}{s_1} \binom{2M-n}{s_2} (-1)^{n-s_1} |0\rangle$$

or

$$M_{n0} = \frac{(-\mathrm{i})^{2n}}{2^{2M}(n!)^{1/2}} \sum_{s_1,s_2} [2\tau(s_1 + s_2 - M)]^n \binom{n}{s_1} \binom{2M-n}{s_2} \times (-1)^{n-s_1} \exp[-2\tau^2(s_1 + s_2 - M)^2]. \tag{29}$$

To derive equation (29), we use the following relations:

$$\langle n | \exp(\alpha a^+) \exp(\alpha a) |0\rangle = \langle n | \exp(\alpha a^+) |0\rangle = \alpha^n/(n!)^{1/2}. \tag{30}$$

From equations (21) and (29), we find that

$$p(n, \tau) = (-\mathrm{i})^{2n} \exp(-\mathrm{i}\omega_1 M\tau)(n!)^{-3/2} 2^{-2M} \left[\prod_{i=1}^{n} (r + M - \mathrm{i} + 1)(r - M + \mathrm{i}) \right]^{1/2} \times \sum_{s_1,s_2} \left\{ \binom{n}{s_1} \binom{2M-n}{s_2} (-1)^{n-s_1} [2\tau(s_1 + s_2 - M)]^n \times \exp[-2\tau^2(s_1 + s_2 - M)^2] \right\}. \tag{31}$$

The result obtained in equation (31) is not general. Since we have assumed $2M \geqslant n$, and considering that $n_{\max} = M + r$, this condition becomes

$$M \geqslant r. \tag{32}$$

The relation (32) can be satisfied with the equal sign for all times only in the case $M = r$. In all other cases the solution is valid for short times, before reaching $n_{\max}$ so that the relation $2M \geqslant n$ is satisfied. Beyond that time, this solution (equation (31)) becomes invalid.

(*b*) General solution (spontaneous emission). We can write M_{n0} in the form

$$M_{n0} = (-i)^n \langle n| \exp[n \ln \tan \tau(a + a^+) + 2M \cos \tau(a + a^+)]|0\rangle \tag{33}$$

and expand $\ln \tan \tau(a + a^+)$ and $\ln \cos \tau(a + a^+)$ in power series in the argument $\tau(a + a^+)$ (Abramowitz and Stegun 1970). It is simple to prove that (Appendix A)

$$M_{n0} = (-\mathrm{i})^n \tau^n \langle n|(a + a^+)^n \prod_{p=1}^{\infty} \exp[c_p(a + a^+)^{2p}]|0\rangle \tag{34}$$

with

$$\begin{aligned} c_p &= na_p + 2Mb_p, \\ a_p &= (-1)^{p-1}2^{2p}(2^{2p-1} - 1)B_{2p}\tau^{2p}/(p)(2p)!, \\ b_p &= (-1)^p 2^{2p-1}(2^{2p} - 1)B_{2p}\tau^{2p}/(p)(2p)!, \end{aligned} \tag{35}$$

where B_{2p} are the Bernoulli numbers, defined as (Abramowitz and Stegun 1970)

$$B_{2p} = [(-1)^{p-1}(2)(2p)!/(2\pi)^{2p}] \sum_{K=1}^{\infty} K^{-2p}. \tag{36}$$

Expanding the exponential in equation (34), and setting

$$\prod_{p=1}^{\infty} \sum_{q=0}^{\infty} (c_p)^q (a + a^+)^{2pq}/q! = \sum_{v=0}^{\infty} d_v (a + a^+)^{2v}, \tag{37}$$

equation (34) becomes

$$M_{n0} = (-\mathrm{i})^n \tau^n \langle n| \sum_{v=0}^{\infty} d_v (a + a^+)^{2v+n}|0\rangle \tag{38}$$

or

$$M_{n0} = (-\mathrm{i})^n \tau^n \sum_{v=0}^{\infty} d_v \langle n|(a + a^+)^{2v+n}|0\rangle. \tag{39}$$

After some straightforward calculation (Appendix B) we obtain

$$M_{n0} = ((-\mathrm{i})^n \tau^n/(n!)^{1/2}) \sum_{v=0}^{\infty} \frac{(2v + n)!}{2^v v!} d_v \tag{40}$$

and

$$p(n, \tau) = [(-\mathrm{i})^n \tau^n \exp(-\mathrm{i}\omega_1 M\tau)/(n!)^{3/2}] \times \left[\prod_{q=1}^{n} (r + M - q + 1)(r - M + q)\right]^{1/2} \sum_{v=0}^{\infty} \frac{(2v + n)!}{2^v v!} d_v. \tag{41}$$

From the definition of d_v (equation (37)), we can write

$$d_v = \left[\prod_{p=1}^{\infty} \sum_{q=0}^{\infty} (c_p)^q/q!\right]_{\Sigma_{pq}=v}. \tag{42}$$

To visualise the rather complicated sum and product appearing in equation (42), we list here the first five terms:

$$d_0 = 1 \qquad d_1 = \frac{(c_1)^1}{1!} \qquad d_2 = \frac{(c_1)^2}{2!} + \frac{(c_2)^1}{1!}$$

$$d_3 = \frac{(c_1)^3}{3!} + \frac{(c_1)^1(c_2)^1}{(1!)(1!)} + \frac{(c_3)^1}{1!}$$

$$d_4 = \frac{(c_1)^4}{4!} + \left[\frac{(c_1)^2}{2!}\right]\left[\frac{(c_2)^1}{1!}\right] + \frac{(c_2)^2}{2!} + \frac{(c_4)^1}{1!} + \frac{(c_1)^1}{1!}\frac{(c_3)^1}{1!}.$$

Since the product pq in equation (42) cannot exceed v, the upper limits of the sum and product will be v rather than infinity, and $p(n, \tau)$ becomes

$$p(n,\tau) = (-\mathrm{i})^n \tau^n \exp(-\mathrm{i}\omega_1 M\tau)(n!)^{-1/2}\left[\prod_{q=1}^{n} (r+M-q+1)(r-M+q)\right]^{1/2}$$
$$\times \sum_{v=1}^{\infty} \frac{(2v+n)!}{2^v v!\, n!}\left\{n! + \left[\prod_{p=1}^{v} \sum_{\omega=1}^{v} \frac{(c_p)^\omega}{\omega!}\right]_{\Sigma_{p\omega}=v}\right\} \tag{43}$$

or, in a final form

$$p(n,\tau) = (-\mathrm{i})^n \tau^n \exp(-\mathrm{i}\omega_1 M\tau)(n!)^{-1/2}\left[\prod_{q=1}^{n} (r+M-q+1)(r-M+q)\right]^{1/2}$$
$$\times\left\{1 + \frac{1}{n!}\sum_{v=1}^{\infty}\left[\frac{(2v+n)!}{v!}(-2\tau^2)^v \sum_{\{n_i\}} \frac{(c_1^*)^{n_1}(c_2^*)^{n_2}\dots(c_v^*)^{n_v}}{n_1!\, n_2! \dots n_v!}\right]\right\} \tag{44}$$

where

$$c_p^* = [B_{2p}/(p)(2p)!][(n-M) - 2^{p-1}(n-2M)] \tag{45}$$

and by $\Sigma_{\{n_i\}}$ is meant $\Sigma_{n_1+2n_2+3n_3+\dots+vn_v=v}$.

Using the definition of the Bernoulli numbers (equation (36)), c_p^* can be expressed as follows:

$$c_p^* = \frac{(-1)^{p-1}}{p}\left(\sum_{K=1}^{\infty} (2\pi K)^{-2p}\right)[2(n-M) - 2^p(n-2M)]. \tag{46}$$

Finally, since equation (29) is not exact (a detailed discussion is presented at the end of this paper), the normalised probability $|p(n,\tau)|^2$ becomes

$$|p(n,\tau)|^2 = |p(n,\tau)|^2_{\mathrm{eqn}(43)} \Big/ \sum_{n=0}^{n_{\max}} |p(n,\tau)|^2_{\mathrm{eqn}(43)}. \tag{46a}$$

The same argument applies to the result given in equation (31). Summarising this section, the results are

$$|p(n,\tau)|^2=\prod_{i=1}^{n}(r+M-\mathrm{i}+1)(r-M+\mathrm{i})$$

$$\times\left|\sum_{s_1;s_2}\binom{n}{s_1}\binom{2M-n}{s_2}(-1)^{n-s_1}[2\tau(s_1+s_2-M)]^n\exp[-2\tau^2(s_1+s_2-M)^2]\right|^2$$

$$\times\left(2^{-4M}\Big/(n!)^3\sum_n|p(n,\tau)|^2\right) \tag{46b}$$

for the case $2M\geq n$, and for the general case (spontaneous emission)

$$|p(n,\tau)|^2=(\tau)^{2n}\prod_{q=1}^{n}(r+M-q+1)(r-M+q)$$

$$\times\left|\sum_{v=1}^{\infty}\frac{(2v+n)!}{2^v v!n!}\left[n!+\left(\prod_{p=1}^{v}\sum_{\omega=1}^{v}\frac{(c_p)^{\omega}}{\omega!}\right)\right]_{\Sigma p\omega=v}\right|^2\left(n!\sum_{n=0}^{n_{\max}}|p(n,\tau)|^2\right)^{-1}. \tag{46c}$$

4. Particular cases

(*a*) Short time solution, $n=1$. From equation (44), neglecting the sum over v, which involves a power series expansion in τ^2, we obtain

$$|p(1,\tau)|^2\approx\tau^2(r+M)(r-M+1) \tag{47}$$

which is Dicke's result, using first order perturbation theory (Dicke 1954). An identical result is obtained when the limit $\tau\sim$ small is taken in equation (31).

(*b*) $M=\frac{1}{2}$, $n=1$, $r=\frac{1}{2}$. This case was solved exactly in the rotating-wave approximation by Jaynes and Cummings (1963, also Allen and Eberly 1975, Louisell 1964). At resonance, their result ($n(0)=0$) is

$$|p(1,\tau)|^2=\sin^2\tau. \tag{48}$$

Using equation (31) for the values of $M=\frac{1}{2}$, $n=1$, $r=\frac{1}{2}$, the summation over s_2 disappears ($s_2=0$) and we obtain (for the normalised probability)

$$|p(1,\tau)|^2_{\text{normalised}}=\tau^2/(1+\tau^2). \tag{49}$$

If we now use the expression given by equation (44), the result obtained is identical to equation (49).

The average number of photons is given by

$$\bar{n}=\tau^2/(1+\tau^2). \tag{50}$$

Notice that for

$$\tau\to\infty,\qquad \bar{n}\to 1.$$

The difference between our solution and the result of Jaynes and Cummings is discussed at the end of this paper.

5. The solution (stimulated emission)

Let us assume that the initial state of the system is characterised by

$$|m(0)=M-n(0)\rangle|n(0)\rangle \tag{51}$$

where the conserved quantity is now

$$M=n(0)+m(0)=n(\tau)+m(\tau). \tag{52}$$

The calculation proceeds as follows: by analogy with equation (16)

$$p(n,\tau)=\exp(-\mathrm{i}\omega_1 M\tau)\langle n|\langle M-n|\exp(y_-R^-)\exp[(\ln y_z)R_3]\exp(y_+R^+)|m(0)\rangle|n(0)\rangle \tag{53}$$

where now $p(n,\tau)$ is interpreted as the probability amplitude for n photons to be emitted at time τ, when initially there were $n(0)$ photons. Since our interest is in the net emission process, $n\geq n(0)$. The relation $n>n(0)$ will be reversed if net absorption is considered. Also, since M is the only conserved quantity, for a fixed value of r there is a limited number of states for which $n<n(0)$ and $r>M>m(0)$. This case is not considered here.

A typical example of $n(0)>n$ will be considered in Orszag (1979b), when the present Hamiltonian is used to describe non-linear optical effects. Both the second harmonic generation and anti-Stokes Raman effect are characterised by $n\leq n(0)$.

Following a procedure similar to the spontaneous emission case, one obtains the following results:

$$\begin{aligned}p(n,\tau)=\sum_{q=0}^{\text{upper } q}&\left\{\prod_{s=1}^{q}[(r-m(0)-s+1)(r+m(0)+s)]^{1/2}\right.\\&\left.\times\prod_{t=1}^{q+\Delta n}[(r+m(0)+q-t+1)(r-m(0)-q+t)]^{1/2}/(q+\Delta n)!\,q!\right\}\\&\times(-\mathrm{i})^{2q+\Delta n}\tau^{2q+\Delta n}\sum_{v=0}^{\infty}\left\{\left(\prod_{p=1}^{v}\sum_{\omega=0}^{v}\frac{(c'_p)^{\omega}}{\omega!}\right)_{\Sigma_{p\omega}=v}\right.\\&\times\sum_{j=0}^{\text{upper } j}\langle[2(q+v)+\Delta n]!\,(n(0)!\,n!)^{1/2}/[2^{q+v-j}(q+v-j)!\\&\left.\times(n(0)-j)!\,j!\,(j+\Delta n)!]\rangle\right\}\exp(-\mathrm{i}\omega_1 M\tau)\end{aligned} \tag{54}$$

where

$$\Delta n=n-n(0),\qquad c'_p=(2q+\Delta n)a_p+2(m(0)+q)b_p \tag{55}$$

and the definitions of a_p and b_p are given by equation (35). This calculation is done in Appendix C. The upper limits of q are $n(0)$ and $n(0)+r-M$ if $M\leq r$ and $M>r$ respectively. The upper limits of j are $n(0)$ and $v+q$ if $n(0)\leq q+v$ and $n(0)>q+v$ respectively.

As a consistency check, if we take $n(0)=0$, the summation over q is reduced to one term ($q=0$), the product $\Pi_s=1$, $\Delta n=n$, $j=0$, $c'_p=c_p$ (equation (35)) and the result of spontaneous emission is regained (equation (41)).

The final normalised probability (stimulated emission) is

$$
\begin{aligned}
|p(n,\tau)|^2 = \Bigg| \sum_{q=0}^{\text{upper } q} & \Bigg\{ \Bigg[\prod_{s=1}^{q} [(r-m(0)-s+1)(r+m(0)+s)] \Bigg]^{1/2} \\
& \times \Bigg[\prod_{t=1}^{q+\Delta n} [(r+m(0)+q-t+1)(r-m(0)-q+t)] \Bigg]^{1/2} \\
& \times (-\mathrm{i})^{2q+\Delta n} [\tau^{2q+\Delta n}/q!\,(q+\Delta n)!] \sum_{v=0}^{\infty} \Bigg[\Bigg(\prod_{p=1}^{v} \sum_{\omega=0}^{v} \frac{(c'_p)^{\omega}}{\omega!} \Bigg)_{\Sigma_{p\omega}=v} \\
& \times \Bigg(\sum_{j=0}^{\text{upper } j} \{(2(q+v)+\Delta n)!\,[n(0)!\,n!]^{1/2}/[2^{q+v-j}(q+v-j)! \\
& \times (n(0)-j)!\,j!\,(j+\Delta n)!]\} \Bigg) \Bigg] \Bigg\} \Bigg|^2 \Bigg(\sum_{n=0}^{n_{\max}} |p(n,\tau)|^2 \Bigg)^{-1}. \qquad (56)
\end{aligned}
$$

6. Discussion

The extension of the unscrambling theorems (equations (9), (10) and (11)) to the case where the coefficients ω_+, ω_- and ω_z are operators is not obvious. Assuming that equations (9), (10) and (11) are true in a two-dimensional representation, it does not follow that the group property of rotations can be applied here since we are not dealing with ordinary rotations. Furthermore, this extension is only valid if the ω's are a set of commuting operators acting on a Hilbert space different from the angular momentum space. The proof is given in Appendix D.

Concerning the time range of validity of the solutions, equation (31) was derived by using power series expansions of the sine and cosine functions; therefore it is valid for all times. However, equations (46) and (54) were derived by using the power-series expansion of ln cos $\tau(a+a^+)$ and ln tan $\tau(a+a^+)$. This expansion is valid provided the argument z of ln tan z is such that

$$|z| < \pi/2. \qquad (57)$$

To estimate the range of validity of the solution we take the classical limit, where the argument z becomes

$$z = 2\alpha\tau \qquad (58)$$

and

$$\alpha \approx n^{1/2}. \qquad (59)$$

The condition of equation (57) thus becomes

$$2n^{1/2}\tau < \pi/2 \qquad (60)$$

which clearly indicates that for a given initial preparation of the system there is a $\tau = \tau_{\max}$ beyond which the solution becomes invalid. In table 1, $\tau_{\max}$ is listed for various values of n.

This limitation applies to the cases when the system has been prepared in the super-radiant state (spontaneous emission) and intermediate states between the super-radiant and complete inversion states (spontaneous and stimulated emission).

Table 1.

n	τ_{max}
$n=0$	no limit
$n=1$	0·7850
$n=2$	0·5553
$n=3$	0·4534
⋮	⋮
$n=10$	0·2483
$n=20$	0·1766

In the case of complete inversion ($M=r$), equation (31) is used and is valid for all times.

The following is a discussion on the validity of the RRWA and comparison with the RWA and exact cases. Let us define

$$\begin{aligned}
\theta_{\text{exact}} &= \exp\{(-\mathrm{i}\omega t)[N+\alpha(a+a^{+})(R^{+}+R)]\},\\
\theta_{\text{RRWA}} &= [\exp(-\mathrm{i}\omega t\hat{N})]\{\exp[(-\mathrm{i}\omega t)\alpha(a+a^{+})(R^{+}+R^{-})]\},\\
\theta_{\text{RWA}} &= [\exp(-\mathrm{i}\omega t\hat{N})]\{\exp[(-\mathrm{i}\omega t)(\alpha)(aR^{+}+a^{+}R^{-})]\},
\end{aligned} \tag{61}$$

where

$$\alpha = K/\omega. \tag{62}$$

In order to make a meaningful comparison we calculate the spontaneous emission for one photon by a power series expansion of the exponentials in equation (61).

A simple calculation shows that

$$\begin{aligned}
\langle 1|\langle M-1|\hat{\theta}_{\text{exact}}|M\rangle|0\rangle \exp(\mathrm{i}\omega tM) &= \alpha C_1+\alpha^3 C_{\text{exact}}+\dots\\
\langle 1|\langle M-1|\hat{\theta}_{\text{RRWA}}|M\rangle|0\rangle \exp(\mathrm{i}\omega tM) &= \alpha C_1+\alpha^3 C_{\text{RRWA}}+\dots\\
\langle 1|\langle M-1|\hat{\theta}_{\text{RWA}}|M\rangle|0\rangle \exp(\mathrm{i}\omega tM) &= \alpha C_1+\alpha^3 C_{\text{RWA}}+\dots
\end{aligned} \tag{63}$$

where

$$\begin{aligned}
C_{\text{exact}} &= \tfrac{1}{6}\mathrm{i}(\omega t)^2\langle M-1|\langle 1|(a+a^{+})^3(R^{+}+R^{-})^3|M\rangle|0\rangle + \text{higher order terms in } (\omega t),\\
C_{\text{RRWA}} &= \tfrac{1}{6}\mathrm{i}(\omega t)^3\langle M-1|\langle 1|(a+a^{+})^3(R^{+}+R^{-})^3|M\rangle|0\rangle,\\
C_{\text{RWA}} &= \tfrac{1}{6}\mathrm{i}(\omega t)^3\langle M-1|\langle 1|(aR^{+}+a^{+}R^{-})^3|M\rangle|0\rangle,\\
C_1 &= -(\mathrm{i}\omega t)[(r+m)(r-M+1)]^{1/2}.
\end{aligned} \tag{64}$$

From equations (63) and (64) we conclude that the difference between the three models arises in the cubic and higher order terms in α. By comparison of C_{exact}, C_{RRWA} and C_{RWA}, we can see that the RRWA model is exact up to $(\alpha)^3(\omega t)^3$ while the RWA neglects part of this term. Therefore the RRWA is a slightly better approximation since it includes more terms from the expansion of the exact model, as compared with the RWA. The main difference between the two models is that in the RWA, only energy conserving terms are considered; for instance, aR^{+} represents the annihilation of a photon and the creation of a unit of energy for the atoms. The model itself leads to periodic interchange of energy between the atoms and the field. Hence one obtains a periodic behaviour for the emitted photons (average).

On the other hand, the RRWA does not take only energy-conserving terms, and therefore, as we can see from the present results, the periodic behaviour is destroyed. Exceptionally, this is not true for the case of one atom, where, as we shall prove, the non-conserving terms do not contribute to the one-photon emission probability.

As far as the normalisation problems arising in the present work are concerned, a careful analysis shows that they are due to the finite spectrum of the $|n\rangle$ states. Because of this fact, equations (29), (40), (B.3) and (C.15) are only approximate. To see this clearly, consider the case $M = r = \frac{1}{2}$. The probability amplitude $p(1, \tau)$ is given by

$$\exp(\mathrm{i}\omega_1 \tau M) p(1, \tau) = \langle -\tfrac{1}{2}|\langle 1| \exp[-\mathrm{i}\tau(a + a^+)(R^+ + R^-)]|0\rangle|\tfrac{1}{2}\rangle, \tag{65}$$

which can be expanded in a power series as follows:

$$\exp(\mathrm{i}\omega_1 \tau M) p(1, \tau) = \langle -\tfrac{1}{2}|\langle 1|(1 - \mathrm{i}\tau(a + a^+)(R^+ + R^-) - (\tau^2/2!)(a + a^+)^2(R^+ + R^-)^2 \ldots |0\rangle|\tfrac{1}{2}\rangle. \tag{66}$$

The Hilbert space of the $|n\rangle$ states is truncated to two states, $|0\rangle$ and $|1\rangle$. Using the well known properties of the R^+ and R^- operators one obtains

$$\exp(\mathrm{i}\omega_1 M\tau) p(1, \tau) = (-\mathrm{i})(\tau - (\tau^3/3!) + (\tau^5/5!) \ldots) \tag{67}$$

and

$$|p(1, \tau)|^2 = \sin^2 \tau. \tag{68}$$

Similarly we obtain

$$|p(0, \tau)|^2 = \cos^2 \tau, \tag{69}$$

which is in agreement with Jaynes and Cummings. In this special case, as mentioned before, the non-conserving terms did not contribute to the probabilities. In more general terms, one can write

$$\exp(\mathrm{i}\omega_1 \tau M) p(n, \tau) = \langle M - n|\langle n| \exp[-\mathrm{i}\tau(a + a^+)(R^+ + R^-)]|0\rangle|M\rangle \tag{70}$$

for the case of spontaneous emission of n photons. By a power series expansion of equation (70), we obtain

$$\exp(\mathrm{i}\omega_1 M\tau) p(n, \tau) = \sum_{m=0}^{\infty} \frac{(-\mathrm{i}\tau)^m}{m!} \{\langle n|(a + a^+)^m|0\rangle\}\{\langle M - n|(R^+ + R^-)^m|M\rangle\}. \tag{71}$$

Following the techniques used in this work, we write

$$\langle n|(a + a^+)^m|0\rangle = (\mathrm{d}^m/\mathrm{d}\beta^m)\langle n| \exp[(a + a^+)\beta]|0\rangle|_{\beta=0} \tag{72}$$

and

$$\langle M - n|(R^+ + R^-)^m|M\rangle = (\mathrm{d}^m/\mathrm{d}\beta^m)\langle M - n| \exp[(R^+ + R^-)\beta]|M\rangle|_{\beta=0}. \tag{73}$$

Since the angular momentum operators act on a finite spectrum there is no difficulty in using the unscrambling theorems for $\exp \beta(R^+ + R^-)$. It is simple to prove that

$$\exp[\beta(R^+ + R^-)] = \exp[(\tanh \beta)R^-] \exp[\ln(\cosh^2 \beta)R_3] \exp[(\tanh \beta)R^+] \tag{74}$$

and, using equation (74), the following result is readily obtained:

$$\langle M-n|(R^+ + R^-)^m|M\rangle = A\left|\frac{\mathrm{d}^m}{\mathrm{d}\beta^m}[(\cosh\beta)^M)(\tanh\beta)^n]\right|_{\beta=0} \tag{75}$$

with

$$A = \left(\frac{(r+M)!\,(r-M+n)!}{(r-M)!\,(r+M-n)!}\right)^{1/2}. \tag{76}$$

If we now apply a similar technique to the boson operators (equation (72)), using the BCH identity, we obtain

$$\langle n|(a+a^+)^m|0\rangle = (\mathrm{d}^m/\mathrm{d}\beta^m)\langle n|\exp(\beta a^+)|0\rangle \exp(\beta^2/2)|_{\beta=0}$$

$$= \frac{1}{(n!)^{1/2}}(\mathrm{d}^m/\mathrm{d}\beta^m)(\beta^n \exp(\beta^2/2))|_{\beta=0} \tag{77}$$

which can be written in the final form

$$\langle n|(a+a^+)^m|0\rangle = m!/(n!)^{1/2}[\tfrac{1}{2}(m-n)]!\,2^{(m-n)/2}, \tag{78}$$

valid for $m \geqslant n$, and both m and n either even or odd.

Equation (78) is not generally exact. It is exact only for the unbounded spectrum, and in the case of the truncated $|n\rangle$ states, it is exact only up to terms with $m \leqslant M+R$. For $m > M+R$, unwanted states with $n > n_{\max} = M+R$ are generated and computed. To take a simple example, let us return to the one-atom case. A direct computation gives

$$\langle 1|(a+a^+)^3|0\rangle = \langle 1|a^+aa^+ + aa^{+2}|0\rangle \tag{79}$$

which are the only two contributing terms. Since the truncated $|n\rangle$ states contain only the $|0\rangle$ and $|1\rangle$ states the second term in equation (79) generates a $|2\rangle$ state and therefore drops out of the calculation. We obtain

$$\left|\langle 1|(a+a^+)^3|0\rangle\right|_{\substack{\text{truncated}\\ |n\rangle\ \text{states}}} = 1. \tag{80}$$

However, if we assume that both terms contribute (which is obviously incorrect), we obtain

$$\left|\langle 1|(a+a^+)^3|0\rangle\right|_{\substack{\text{unbounded}\\ \text{spectrum}}} = 3. \tag{81}$$

Using equation (78) with $m = 3$ and $n = 1$, we also obtain the result of equation (81), which is incorrect.

The point at issue is the size of the error introduced by using equation (78). For $r = M = \frac{1}{2}$ it is clear that an error is introduced and that the results of Jaynes and Cummings are not regained. This is not a result of the RRWA, as is shown by equations (68) and (69), but is rather due to the inaccuracy of equation (78).

Generally, for a given M and r, and with increasing m, the error will appear only when $m > M+R$ and states with $n > n_{\max}$ are starting to be generated. Therefore an error of the order of $(\tau)^{M+R}$ is introduced. It is clear at this point that since the model itself is only accurate up to α^3, for $(M+R) > 3$, this error is smaller than that inherent in both the RWA and RRWA theories and therefore that it can be neglected.

The previous argument applies to spontaneous emission (§ 3). For stimulated emission, the quantity involved is

$$\langle n|(a+a^+)^m|n(0)\rangle$$

with $n_{\max}=M+R$. The calculations are therefore exact up to

$$m_{\max}=M+R-n(0). \tag{82}$$

Summarising this discussion, the present theory, when applied to a system with a reasonably large number of atoms, is more accurate than the RWA, and the accuracy of the calculations presented here and of the curves presented in Orszag (1979a) are limited only by the accuracy of the model itself.

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I should like to thank both referees who raised some important questions regarding the accuracy and validity of this theory, thus helping me to clarify some of its non-trivial mathematical aspects.

Appendix A. Proof of equation (34)

From Abramowitz and Stegun (1970) we obtain

$$\ln\tan\tau(a+a^+)=\sum_{p=1}^{\infty}a_p(a+a^+)^{2p}+\ln\tau(a+a^+) \tag{A.1}$$

$$\ln\cos\tau(a+a^+)=\sum_{p=1}^{\infty}b_p(a+a^+)^{2p} \tag{A.2}$$

where a_p and b_p are given by equation (35). Using equations (A.1), (A.2), (33) and (35) we obtain

$$M_{n0}=(-\mathrm{i})^n\langle n|\exp\left\{n\left[\sum_{p=1}^{\infty}a_p(a+a^+)^{2p}+\ln\tau(a+a^+)\right]+2M\sum_{p=1}^{\infty}b_p(a+a^+)^{2p}\right\}|0\rangle \tag{A.3}$$

or

$$M_{n0}=(-\mathrm{i})^n\tau^n\langle n|\left[\exp\sum_{p=1}^{\infty}c_p(a+a^+)^{2p}\right](a+a^+)^n|0\rangle \tag{A.4}$$

where c_p is given by equation (35). Equation (34) is the factorised version of equation (A.4).

Appendix B. Proof of equation (40)

Using the BCH formula (34) we obtain

$$\langle n|(a+a^+)^{2v+n}|0\rangle=(\mathrm{d}^{2v+n}/\mathrm{d}\epsilon^{2v+n})\{\exp(\epsilon^2/2)\langle n|\exp(\epsilon a^+)\exp(\epsilon a)|0\rangle\}_{\epsilon=0} \tag{B.1}$$

or

$$\langle n|(a+a^+)^{2v+n}|0\rangle = \frac{1}{(n!)^{1/2}} \frac{\mathrm{d}^{2v+n}}{\mathrm{d}\epsilon^{2v+n}} \left(\sum_{m=0}^{\infty} \frac{\epsilon^{2m+n}}{2^m m!} \right)_{\epsilon=0} \tag{B.2}$$

$$\langle n|(a+a^+)^{2v+n}|0\rangle = (2v+n)![(n!)^{1/2} 2^v v!]. \tag{B.3}$$

Substituting equation (B.3) into equation (39), we obtain equation (40).

Appendix C. Proof of equation (54)

$$p(n,\tau) = \exp(-\mathrm{i}\omega_1 M\tau)\langle n|\langle M-n|\exp(y_- R^-)\exp[(\ln y_z)R_3]\exp(y_+ R^+)|m(0)\rangle|n(0)\rangle. \tag{C.1}$$

If

$$\begin{aligned} M &= m(0)+n(0) > r, \quad m(0)_{\max} = r \\ M &= m(0)+n(0) \leqslant r, \quad m(0)_{\max} = M. \end{aligned} \tag{C.2}$$

Considering the condition (C.2) $\exp y_+ R^+$ can be expanded in a power series of the argument:

$$\exp(y_+ R^+)|m(0)\rangle = \sum_{q=0}^{\text{upper } q} [(y_+)^q (R^+)^q / q!]|m(0)\rangle. \tag{C.3}$$

The upper limit of q is $n(0)$ if $M \leqslant r$ and $n(0)+r-M$ if $M > r$. From the well-known rules for angular momentum operators we have

$$(R^+)^q|m(0)\rangle = \prod_{s=1}^{q} [(r-m(0)-s+1)(r+m(0)+s)]^{1/2}|m(0)+q\rangle$$

and

$$\exp[(\ln y_z)R_3]|m(0)+q\rangle = (y_z)^{m(0)+q}|m(0)+q\rangle. \tag{C.4}$$

In equation (C.4), it is understood that $\Pi_{s=1}^{q} = 1$ when $q=0$. Substituting (C.3) and (C.4) into (C.1), we obtain

$$\begin{aligned} p(n,\tau) = \exp(-\mathrm{i}\omega_1 M\tau)\langle n|\langle M-n|\exp(y_- R^-) \sum_{q=0}^{\text{upper } q} \Bigg[\frac{(y_+)^q (y_z)^{q+m(0)}}{q!} \\ \times \prod_{s=1}^{q} [(r-m(0)-s+1)(r+m(0)+s)]^{1/2} \Bigg] |m(0)+q\rangle|n(0)\rangle. \end{aligned} \tag{C.5}$$

Let us define

$$A_s \equiv [(r-m(0)-s+1)(r+m(0)+s)]^{1/2}, \tag{C.6}$$

and it follows from the angular momentum rules that if we expand

$$\exp(y_- R^-) = \sum_{u=0}^{\infty} (y_-)^u (R^-)^u / u!, \tag{C.7}$$

then

$$\langle M-n|(R^-)^u|m(0)+q\rangle = \prod_{t=1}^{u} (B_t)\delta_{u,q+\Delta n} \tag{C.8}$$

where

$$B_t \equiv [(r+m(0)+q-t+1)(r-m(0)-q+t)]^{1/2}.$$

If we substitute (C.6) and (C.7) in (C.5) and use the condition (C.8), we obtain

$$p(n,\tau)=\sum_{q=0}^{\text{upper } q}\left[\left(\prod_{s=1}^{q} A_s\right)\left(\prod_{t=1}^{q+\Delta n} B_t\right)\Big/(q+\Delta n)!\,q!\right]$$
$$\times\langle n|(y_-)^{q+\Delta n}(y_+)^q(y_z)^{m(0)+q}|n(0)\rangle\exp(-\mathrm{i}\omega_1 M\tau). \qquad \text{(C.9)}$$

At resonance

$$y_+=y_-=-\mathrm{i}\tan\tau(a+a^+)$$
$$y_z=\cos^2\tau(a+a^+), \qquad \text{(C.10)}$$

and

$$M_{n,n(0)}=\langle n|(y_-)^{q+\Delta n}(y_+)^q(y_z)^{q+m(0)}|n(0)\rangle,$$

$$M_{n,n(0)}=\langle n|\exp[(2q+\Delta n)\ln\tan\tau(a+a^+)$$
$$+2(m(0)+q)\ln\cos\tau(a+a^+)]|n(0)\rangle(-\mathrm{i})^{2q+\Delta n}. \qquad \text{(C.11)}$$

From (A.3), (A.4) and (A.5) we can write

$$M_{n,n(0)}=(-\mathrm{i})^{2q+\Delta n}\langle n|\tau^{2q+\Delta n}(a+a^+)^{2q+\Delta n}\prod_{p=1}^{\infty}\sum_{\omega=0}^{\infty}\frac{(c_p')^{\omega}(a+a^+)^{2p\omega}}{\omega!}|n(0)\rangle. \qquad \text{(C.12)}$$

Defining

$$\prod_{p=1}^{\infty}\sum_{\omega=0}^{\infty}\frac{(c_p')^{\omega}(a+a^+)^{2p\omega}}{\omega!}=\sum_{v=0}^{\infty}d_v(a+a^+)^{2v}, \qquad \text{(C.13)}$$

it follows that

$$M_{n,n(0)}=(-\mathrm{i})^{2q+\Delta n}\tau^{2q+\Delta n}\sum_{v=0}^{\infty}[d_v\langle n|(a+a^+)^{2v+2q+\Delta n}|n(0)\rangle]. \qquad \text{(C.14)}$$

Calculation of $N_{n,n(0)}=\langle n|(a+a^+)^{2v+2q+\Delta n}|n(0)\rangle$:

$$N_{n,n(0)}=\frac{\mathrm{d}^{2v+2q+\Delta n}}{\mathrm{d}\epsilon^{2v+2q+\Delta n}}[\langle n|\exp[\epsilon(a+a^+)]|n(0)\rangle]_{\epsilon=0}$$
$$=\frac{\mathrm{d}^{2v+2q+\Delta n}}{\mathrm{d}\epsilon^{2v+2q+\Delta n}}[\exp(\epsilon^2/2)\langle n|\exp(\epsilon a^+)\exp(\epsilon a)|n(0)\rangle]_{\epsilon=0}$$

(in the last step the BCH formula was used)

$$N_{n,n(0)}=\frac{\mathrm{d}^{2v+2q+\Delta n}}{\mathrm{d}\epsilon^{2v+2q+\Delta n}}\left(\exp(\epsilon^2/2)\langle n|\sum_{j=0}^{\infty}\frac{\epsilon^{2j+\Delta n}}{(j+\Delta n)!\,j!}(a^+)^{j+\Delta n}(a)^j|n(0)\rangle\right)$$
$$=\frac{\mathrm{d}^{2v+2q+\Delta n}}{\mathrm{d}\epsilon^{2v+2q+\Delta n}}\left(\exp(\epsilon^2/2)\sum_{j=0}^{\text{upper } j}\epsilon^{2j+\Delta n}(n(0)!\,n!)^{1/2}/(n(0)-j)!\,(j+\Delta n)!\,j!\right)_{\epsilon=0}. \qquad \text{(C.15)}$$

In equation (C.15) the upper limit of j is $n(0)$ rather than infinite, since

$$a^j|n(0)\rangle=0 \qquad \text{if } j>n(0).$$

Proceeding with the calculation of $N_{n,n(0)}$, we write

$$N_{n,n(0)}=\frac{\mathrm{d}^{2v+2q+\Delta n}}{\mathrm{d}\epsilon^{2v+2q+\Delta n}}\left(\sum_{k=0}^{\infty}\sum_{j=0}^{\text{upper } j}\frac{\epsilon^{2j+2k+\Delta n}(n(0)!\,n!)^{1/2}}{j!\,k!\,2^{k}(j+\Delta n)!\,(n(0)-j)!}\right)_{\epsilon=0} \tag{C.16}$$

$$N_{n,n(0)}=\sum_{j=0}^{\text{upper } j}\frac{(2(v+q)+\Delta n)!\,(n(0)!\,n!)^{1/2}}{2^{v+q-j}(v+q-j)!\,j!\,(j+\Delta n)!\,(n(0)-j)!}. \tag{C.17}$$

The upper limit of j is

$$\begin{array}{ll} n(0) & \text{if } (q+v)\geqslant n(0) \\ q+v & \text{if } (q+v)<n(0). \end{array} \tag{C.18}$$

Finally, we write $M_{n,n(0)}$ as

$$M_{n,n(0)}=(-\mathrm{i})^{2q+\Delta n}\tau^{2q+\Delta n}\sum_{v=0}^{\infty}d_v\sum_{j=0}^{\text{upper } j}\frac{(2(q+v)+\Delta n)!\,(n(0)!\,n!)^{1/2}}{2^{q+v-j}(q+v-j)!\,j!\,(j+\Delta n)!\,(n(0)-j)!} \tag{C.19}$$

and

$$p(n,\tau)=\exp(-\mathrm{i}\omega_1 M\tau)\sum_{q=0}^{\text{upper } q}\left[\left(\prod_{s=1}^{q}A_s\right)\left(\prod_{t=1}^{q+\Delta n}B_t\right)\Big/ q!\,(q+\Delta n)!\right]M_{n,n(0)} \tag{C.20}$$

which is equation (54).

Appendix D. Proof of the extension of the unscrambling theorems in the case when ω_+, ω_-, ω_z are commuting operators.

Assume that

$$\exp(\omega_+J^+ +\omega_-J^- +\omega_zJ_3)=\exp(y_-R^-)\exp[(\ln y_z)R_3]\exp(y_+R^+) \tag{D.1}$$

is true for a two dimensional space, where

$$J^+=\begin{pmatrix}0&1\\0&0\end{pmatrix},\qquad J^-=\begin{pmatrix}0&0\\1&0\end{pmatrix},\qquad J_3=\frac{1}{2}\begin{pmatrix}1&0\\0&-1\end{pmatrix}. \tag{D.2}$$

Also, define the N-particle operators

$$R^+=\sum_{n=1}^{N}(J^+)^{(n)},\qquad R^-=\sum_{n=1}^{N}(J^-)^{(n)},\qquad R_3=\sum_{n=1}^{N}(J_3)^{(n)}, \tag{D.3}$$

where the superscript denotes the specific particle. The corresponding exponential for the N-particle case would be

$$\exp\left(\omega_+\sum_n(J^+)^{(n)}+\omega_-\sum_n(J^-)^{(n)}+\omega_z\sum_n(J_3)^{(n)}\right) \tag{D.4}$$

that can be ordered as follows:

$$\begin{aligned}\exp&\left(\omega_+\sum_n(J^+)^{(n)}+\omega_-\sum_n(J^-)^{(n)}+\omega_z\sum_n(J_3)^{(n)}\right)\\ &=\exp(\omega_+(J^+)^{(1)}+\omega_-(J^-)^{(1)}+\omega_z(J_3)^{(1)})\\ &\quad\times\exp(\omega_+(J^+)^{(2)}+\omega_-(J^-)^{(2)}+\omega_z(J_3)^{(2)})\\ &\quad\times\ldots\times\exp(\omega_+(J^+)^{(n)}+\omega_-(J^-)^{(n)}+\omega_z(J_3)^{(n)}).\end{aligned} \tag{D.5}$$

The last step in equation (D.5) is only valid if ω_+, ω_- and ω_z commute. We also made use of the fact that the angular-momentum operators corresponding to different particles commute with one another. Using equation (D.1) N times, we can write

$$\exp[\omega_+ \sum (J^+)^{(n)} + \omega_- \sum (J^-)^{(n)} + \omega_z \sum (J_3)^{(n)}]$$
$$= \exp[y_-(J^-)^{(1)}] \exp[(\ln y_z)(J_3)^{(1)}] \exp[y_+(J^+)^{(1)}]$$
$$\times \ldots \times \exp[y_-(J^-)^{(n)}] \exp[(\ln y_z)(J_3)^{(n)}] \exp[y_+(J^+)^{(n)}], \qquad \text{(D.6)}$$

and, since y_+, y_- and y_z are functions of ω_+, ω_- and ω_z, they will also be a set of commuting operators; therefore we can shift for instance $\exp y_-(J^-)^{(2)}$ in equation (D.6) to the left, next to $\exp y_-(J^-)^{(1)}$ and we can continue the procedure with $\exp y_-(J^-)^{(3)}$, etc. Applying the same method to $\exp(\ln y_z)J_3^i$ and $\exp y_+(J^+)^i$ we get

$$\exp\left(\omega_+ \sum (J^+)^{(n)} + \omega_- \sum (J^-)^{(n)} + \omega_z \sum (J_3)^{(n)}\right)$$
$$= \exp\left[y_- \sum (J^-)^{(n)}\right] \exp\left[(\ln y_z) \sum (J_3)^{(n)}\right] \exp\left[y_+ \sum (J^+)^{(n)}\right] \qquad \text{(D.7)}$$

which is what we wanted to prove.

As for the proof of equation (D.1) for the two-dimensional cases, we can follow a procedure identical to Arecchi *et al* (1972), the only difference being that in the Maclaurin expansion the coefficients are operators rather than c-numbers, again provided that ω_+, ω_- and ω_z commute.

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